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On The Index of a Holomorphic Vector Field Tangent to a Singular Variety

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Abstract. In this article we define and compute an index for a holomorphic vector field on a (possibly singular) subvariety of a complex manifold, provided the subvariety is a local complete intersection. This index reduces to the usual Poincaré-Hopf index in case the subvariety is smooth, and is equal more generally to the index defined in [**GSV**] and [**SS**].

1. Introduction

In [**GSV**], X. Gomez-Mont, J. Seade and A. Verjovsky defined a "topological" index $\mathrm{Ind}_{m_0}(X, V)$ for a holomorphic vector field X on a complex hypersurface V with an isolated singularity m_0 in a complex manifold W, generalizing the usual Poincaré-Hopf index of the smooth case (see also C. Bonatti and X. Gomez-Mont [**BG**]). In [**G**], X. Gomez-Mont defined a "homological" index in the similar situation when V is a subvariety of arbitrary codimension in W, and proved that it coincides with the previous one when the variety is a hypersurface, providing a computation of it in terms of local algebra. In [**SS**], J. Seade and the third author gave a formula for computing a similar index by desingularization, for higher codimensional V which are complete intersections.

We define here another index (call it "differential" index), also in arbitrary codimension, by means of an integral formula and prove that this differential index still coincides with the previous one. In fact, we define more generally the "residues" of a holomorphic vector field on a singular V, which is a local complete intersection, generalizing the formula of [$\mathbf{BB}_1$] and [$\mathbf{BB}_2$] expressing, when V is smooth, the residues

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(including the Poincaré-Hopf index) as Grothendieck residues.

More precisely, we get the following result. Let X be a holomorphic vector field on a complex manifold W of complex dimension n, tangent to a (possibly singular) subvariety V of complex dimension p. We shall assume furthermore that V is a "strong" local complete intersection (SLCI), which signifies (see [**LS**] section 2) in particular that the normal bundle to the regular part of V in W has a C^∞ extension $\tilde{v}_V$ to some open neighborhood U of V in W (the restriction of which to V, denoted by v_V, being natural). Note that smooth subvarieties (submanifolds), hypersurfaces and complete intersections are all SLCI's. Let m_0 be an isolated point of $\mathrm{Sing}(X) \cap V$, U_0 a neighborhood of m_0 in W and

$$f = (f_1, \dots, f_q): U_0 \to \mathbb{C}^q, q = n - p,$$

a holomorphic map such that $V \cap U_0 = f^{-1}(0)$, the ideal generated by the components f_u being assumed to be reduced. Let C be the $q \times q$ matrix with holomorphic coefficients such that $X \cdot f = \langle C, f \rangle$. Assume that $(z_1, \dots, z_n)$ is a system of complex coordinates on U_0 such that, when we write X as

$$\sum_{i=1}^{n} A_i \frac{\partial}{\partial z_i},$$

the sequence $(A_1, \dots, A_{n-q}, f_1, \dots, f_q)$ is regular, thus the $n - q$ open sets $A_i \neq O (1 \leq i \leq n-1)$ cover completely $(V \cap U_0) - m_0$ (such a coordinate system always exists, see Theorem 2 of [**LS**]).

Let J denote the jacobian matrix

$$\frac{D(A_1, \dots, A_n)}{D(z_1, \dots, z_n)}.$$

We also denote by $[c(-J) \cdot c(-C)^{-1}]_k$ the holomorphic function given as the coefficient of t^k in the formal power series expansion of

$$\det\left(I_n - t\frac{\sqrt{-1}}{2\pi}J\right) \cdot \left[\det\left(I_q - t\frac{\sqrt{-1}}{2\pi}C\right)\right]^{-1}$$

in t, where I_n and I_q denote the identity matrices of sizes n and q. Then we have

Theorem 1. *Define the index* $\mathrm{Ind}_{V,m_0}(X)$ *of* X *(on* V*) at* m_0 *by*

$$\mathrm{Ind}_{V,m_0}(X) = \int_R \frac{[c(-J)\cdot c(-C)^{-1}]_{n-q} dz_1 \wedge \ldots \wedge dz_{n-q}}{\prod_{i=1}^{n-q} A_i},$$

where R *denotes the set* $\{f = 0, |A_i| = \varepsilon, 1 \le i \le n-q\}$, *for some small* $\varepsilon > 0$, *the real hypersurfaces* $|A_i| = \varepsilon$ *being assumed to be in general position and* R *being oriented so that*

$$d(\arg A_1) \wedge d(\arg A_2) \wedge \ldots \wedge d(\arg A_{n-q})$$

is positive. Then

(i) *The above integral is an integer, in fact it coincides with the topological index of [***GSV***] and [***SS***].*

(ii) *If* V *is compact and if all points* m_α *of* $\mathrm{Sing}(X)\cap V$ *are isolated, then the sum* $\sum_\alpha \mathrm{Ind}_{V,m_\alpha}(X)$ *is equal to the Chern number*

$$\langle c_{n-q}(T(W)|_{V-\nu_V}), V\rangle$$

of the virtual bundle $[T(W)|_{V-\nu_V}]$ *tangent to* V, ν_V *denoting the natural extension to* V *of the normal bundle to the regular part of* V *in* W.

Index for a "non-degenerate" vector field

We further assume that, in terms of the above coordinate system $(z_1, \ldots, z_n)$, the functions $A_1, \ldots, A_{n-q}$ depend only on $(z_1, \ldots, z_{n-q})$ and that

$$\det J_{n-q}(m_0) \neq 0, \quad J_{n-q} = \frac{D(A_1, \ldots, A_{n-q})}{D(z_1, \ldots, z_{n-q})}.$$

Thus we have $dA_1 \wedge \ldots \wedge dA_{n-q} = \det J_{n-q} dz_1 \wedge \ldots \wedge dz_{n-q}$ and we may choose $(A_1, \ldots, A_{n-q}, z_{n-q+1}, \ldots, z_n)$ as a coordinate system near m_0. Denote by $(\lambda_1, \ldots, \lambda_{n-q})$, $(\lambda_1, \ldots, \lambda_{n-q+1}, \ldots, \lambda_n)$ and $(\mu_1, \ldots, \mu_q)$ the eigenvalues of $J_{n-q}(m_0)$, $J(m_0)$ and $C(m_0)$, respectively. Note that $\lambda_1, \ldots, \lambda_{n-q}$ are all non-zero, while some of $\lambda_{n-q+1}, \ldots, \lambda_n, \mu_1, \ldots, \mu_q$ may be zero. Setting

$$P_J(x_1, \ldots, x_n) = \prod_{i=1}^{n}(1+x_i), \quad P_C(y_1, \ldots, y_q) = \prod_{j=1}^{q}(1+y_j),$$

we denote by

$$\left[\frac{P_J(x_1,\dots,x_n)}{P_C(y_1,\dots,y_n)}\right]_k$$

the homogeneous component of degree k (with respect to the $n+q$ variables $(x_1,\dots,x_n,y_1,\dots,y_q)$) in the quotient

$$\frac{P_J(x_1,\dots,x_n)}{P_C(y_1,\dots,y_n)}.$$

Then, if V is represented as an ℓ-fold covering of the $(A_1,\dots,A_{n-q})$ space, the integral in Theorem 1 is computed as

$$\mathrm{Ind}_{V,m_0}(X) = \left[\frac{P_J(\lambda_1,\dots,\lambda_n)}{P_C(\mu_1,\dots,\mu_q)}\right]_{n-q} \cdot \frac{\ell}{\prod_{i=1}^{n-q}\lambda_i}.$$

Especially, for $q=1$ (V: hypersurface), we get

$$\mathrm{Ind}_{V,m_0}(X) = \frac{1}{\mu_1}\left(\prod_{i=1}^{n}\lambda_i - \prod_{i=1}^{n}(\lambda_i-\mu_i)\right)\frac{\ell}{\prod_{i=1}^{n-1}\lambda_i}.$$

The proof of Theorem 1 is done in two steps; first extending Theorem 1' of [**LS**] to virtual bundles (in particular to $T(W)|_{V-\nu_V}$), whose restriction to the regular part of V admit a natural action of X, and then proving, by means of "smoothing", that the residue corresponding to c_{n-q} is equal to the topological index of [**GSV**] and [**SS**]. On the way, we shall get similar residues for an arbitrary characteristic class of this virtual tangent bundle, in dimension $2(n-q)$, see Theorem 2 below.

2. Extension of Bott residues to virtual bundles on singular varieties

In this section, we sketch briefly how to extend Theorem 1' of [**LS**] to the case of virtual bundles. Let us just remark that the situation that we are going to study is not the same as for the residues of virtual bundles considered by Baum and Bott in [**BB**$_1$] and [**BB**$_2$]; first, the base space V of the bundles may be here a *singular* variety and secondly each component of the virtual bundles that we have in mind here is equipped with an action of a holomorphic vector field (in a sense precised below, in the spirit of Bott [**BB**$_2$]), which was not true in general for the virtual normal bundle associated to a singular foliation as defined by Baum and Bott (except when the bundle tangent to the singular foliation could be extended as a trivial bundle).

Let W be a complex manifold of complex dimension n and V an analytic subvariety (not necessarily everywhere smooth) of W of complex dimension p. Also, let X be a holomorphic vector field (with singularities) on the regular part of V and write $\sum = \mathrm{Sing}(X) \cup \mathrm{Sing}(V)$. (Recall that a singular point of X is either a point where X is not defined or a point where it vanishes.)

A complex vector bundle $E \to V$ will be said an "X-bundle", if

(i) The restriction of E to the regular part of V is holomorphic,

(ii) the restriction of E to $V - \Sigma$ is equipped with an X action (in the sense of Bott [**B**$_2$]), i.e., a $\mathbb{C}$-linear endomorphism θ_X of the space of C^∞ sections of $E|_{V-\Sigma}$ such that $\theta_X(\sigma)$ is holomorphic for any holomorphic section σ, and that $\theta_X(u\sigma) = (X \cdot u)\sigma + u\theta_X(\sigma)$, for any function u and a section σ,

(iii) either V is smooth, or there exists a C^∞ extension $\tilde{E}$ of the bundle E to some open neighbourhood U of V in W.

Let (E, θ_X) be an X-bundle and ω a connexion on $E|_{V-\Sigma}$. Such a connection will be said *special* relative to θ_X, if its derivation law ∇ satisfies:

$$\begin{cases} \nabla_X \sigma = \theta_X \sigma \text{ for every } C^\infty \text{ section } \theta \text{ of } E|_{V-\Sigma}, \\ \nabla_Z \sigma = 0 \text{ for every vector } Z \in T^{0,1}(V - \Sigma) \\ \quad \text{and every holomorphic section } \sigma \text{ of } E|_{V-\Sigma} \end{cases}$$

It is well known, after Bott [**B**$_2$], that such a connection always exists.

We shall make use of the notation Δ_ω for the Chern-Weil homomorphism defined by a connection ω and denote by $\Delta_{\omega_0\omega_1\ldots\omega_r}(\varphi)$ Bott's operator for iterated differences ([**B**$_3$]) so that

$$d \circ \Delta_{\omega_0\omega_1\ldots\omega_r} = \sum_{j=0}^{r} (-1)^j \Delta_{\omega_0\ldots\hat{\omega}_j\ldots\omega_r}.$$

In particular we have

$$d \circ \Delta_{\omega\omega'} = \Delta_{\omega'} - \Delta_{\omega}.$$

Let (E, θ_X) and (E', θ'_X) be two X-bundles on V of respective ranks r and r'. Let $\varphi \in (\mathbb{R}[c_1, \dots, c_r])^{2k}$ and $\varphi' \in (\mathbb{R}[c_1, \dots, c_{r'}])^{2k'}$ be Chern polynomials of respective degrees k and k', defining therefore characteristic classes of dimension $2k$ for E (resp. $2k'$ for E').

Lemma 1 (Relative vanishing lemma). *If $k + k'$ is equal to the complex dimension p of V, and if $\omega_1, \dots, \omega_s$ any $s \geq 0$, (resp. $w'_1, \dots, \omega'_{s'}$, any $s' \geq 0$, $s + s' \geq 1$), denotes a family of connections on $E|_{V-\Sigma}$ (resp. $E'|_{V-\Sigma}$) all special relative to the same θ_X (resp. θ'_X, X being the same), then we have*

$$\Delta_{\omega_1 \dots \omega_s}(\varphi) \wedge \Delta_{\omega'_1} \dots \omega'_{s'}(\varphi') = 0.$$

The proof is an obvious generalization of the "absolute vanishing lemma": this is the name that we shall give to the particular case $k = p$ where E' does not occur. (See [$\mathbf{B}_2$]).

This lemma has in particular for corollary that, for $k + k' = p$, the product $\varphi(E) \smile \varphi'(E')$ of characteristic classes vanishes over $V - \Sigma$. We shall now prove that this product "localizes" near Σ, in the sense that it has a natural lift to $H^{2p}(V, V - \Sigma)$ (giving rise to residues in $H_0(\Sigma)$ by duality when Σ is compact).

Let Σ_0 be a union of connected components of Σ and U_0 an open neighborhood of Σ_0 in W, not intersecting $\Sigma - \Sigma_0$. We assume that Σ_0 is compact and let $\tilde{\mathcal{T}}$ be a compact (real) manifold of dimension $2n$ with boundary in U_0 such that Σ_0 is in the interior of $\tilde{\mathcal{T}}$ and that the boundary $\partial\tilde{\mathcal{T}}$ is transverse to $V - \Sigma_0$. We write $\mathcal{T} = \tilde{\mathcal{T}} \cap V$ and $\partial\mathcal{T} = \partial\tilde{\mathcal{T}} \cap V$.

Let ω_0 be an arbitrary connection on $\tilde{E}|_{U_0}$ (or on $E|_{U_0 \cap V}$ if V is smooth), and ω a connection on $E|_{U_0 \cap V - \Sigma_0}$ special for θ_X. Define also ω'_0 and ω' similarly for (E', θ'_X). We still assume $k + k' = p$ (same notation as for Lemma 1).

Lemma 2 (Lemma of relative residues). *If we define*

$$I_{\Sigma_0}(E, E', \varphi, \varphi', \theta_X, \theta'_X)$$

as being equal to

$$\int_{\mathcal{T}} \Delta_{\omega_0}(\varphi) \wedge \Delta_{\omega'_0}(\varphi') + \int_{\partial\mathcal{T}} [\Delta_{\omega_0\omega}(\varphi) \wedge \Delta_{\omega'_0}(\varphi') + \Delta_{\omega}(\varphi) \wedge \Delta_{\omega'_0\omega'}(\varphi')],$$

then

(i) *This number does not depend on the choices of* $\tilde{\mathcal{T}}$, ω, ω_0, ω', *and* ω'_0.

(ii) *Assume* V *to be compact and let* $(\Sigma_\alpha)_\alpha$ *be the partition of* Σ *into connected components. Then, the number* $\sum_\alpha I_{\Sigma_\alpha}(E, E', \varphi, \varphi', \theta_X, \theta'_X)$ *is equal to the evaluation* $\langle \varphi(E) \cup \varphi'(E'), V\rangle$ *of* $\varphi(E) \cup \varphi'(E')$ *on the fundamental class* $[V]$ *of* V. *In particular, it depends only on* V, E, E', φ *and* φ', *but not on* X, θ_X *and* θ'_X.

(iii) *Assume furthermore that* φ *and* φ' *have integral coefficients with respect to the Chern classes of* E *and* E'. *The previous sum is then an integer.*

Making use of Lemma 1 above, Lemma 2 is an easy generalization of Proposition 2 in [**LS**] (corresponding to the absolute case, V being singular). For $k = p$ and $\varphi' = 1$, i.e. in the absolute case, we shall write $I_{\Sigma_0}(E, \varphi, \theta_X)$ instead of $I_{\Sigma_0}(E, E', \varphi, 1, \theta_X, \theta'_X)$.

As usual, denoting by $c(E)$ and $c(E')$ the total Chern classes of E and E', we define the Chern classes $c_j(E - E')$ of the virtual bundle $E - E'$ as being equal to the homogeneous components of the formal expansion of $c(E) \cdot (c(E'))^{-1}$. So we may also define the Chern number of $\varphi''(E - E')$ as the sum of numbers of the form $\langle \varphi(E) \cup \varphi'(E'), V\rangle$, where $k + k' = p$, hence the definition of $I_{\Sigma_\alpha}(E, E', \varphi'', \theta_x, \theta'_X)$.

If V is smooth and if there exists an exact sequence of holomorphic vector bundles

$$0 \to E'' \to E \to E' \to 0$$

on V, then the Chern numbers of E'' and those of $E - E'$ are obviously the same. Assume furthermore that E'' is also an X-bundle with action θ''_X. If θ''_X is equal to the restriction of θ_X to the sections of E'' and if

θ'_X is defined from θ_X by passing to the quotient, it is then less obvious, but true, that the residues of E'' and those of $E - E'$ are also the same:

Lemma 3 (Lemma of residues for Whitney sums). *We have*

$$I_{\Sigma_\alpha}(E'', \varphi'', \theta''_X) = \sum_j I_{\Sigma_\alpha}(E, E', \varphi_j, \varphi'_j, \theta_X, \theta'_X),$$

where $\varphi'' \in (\mathbb{R}[c_1, \ldots, c_{r''}])^{2p}$ *is a Chern polynomial of degree* p *giving rise to a Chern number of* E'', $r'' = r - r'$ *denotes the rank of* E'', *while* $\sum_j \varphi_j \cdot \varphi'_j$ *denotes the image of* φ'' *by the morphism*

$$\mathbb{R}[c''_1, \ldots, c''_{r''}] \to \mathbb{R}[c_1, \ldots, c_r] \otimes \mathbb{R}[c'_1, \ldots, c'_{r'}]$$

generated by the map

$$1 + c''_1 + \cdots + c''_{r''} \mapsto (1 + c_1 + \cdots + c_r) \cdot (1 + c'_1 + \cdots + c'_{r'})^{-1}$$

(with the terms of degree greater than n *truncated).*

In fact, the lemma is proved by taking connections ω'', ω and ω' on E'', E and E', respectively, away from Σ_α so that they are special relative to θ''_X, θ_X and θ'_X, respectively, and that the family $(\omega'', \omega, \omega')$ is compatible with the above exact sequence ([**BB**$_2$] Definition (4.16)). The existence of such a family of connections is not difficult to see.

We will now generalize Theorem 1′ of [**LS**] to the case of virtual bundles. We denote by Σ and $(\Sigma_\alpha)_\alpha$, respectively, the singular set $\Sigma = \mathrm{Sing}(X) \cup \mathrm{Sing}(V)$ and its connected components, as before.

Assume that Σ_α is compact and let U_α be an open neighbourhood of Σ_α in W such that $V_\alpha - \Sigma_\alpha$ is in the regular part of V, $V_\alpha = V \cap U_\alpha$. Let $\tilde{\mathcal{T}}_\alpha$ be a compact real manifold of dimension $2n$ with boundary in U_α such that Σ_α is in the interior of $\tilde{\mathcal{T}}_\alpha$ and that the boundary $\partial\tilde{\mathcal{T}}_\alpha$ is transverse to $V - \Sigma$. We write $\mathcal{T}_\alpha = \tilde{\mathcal{T}}_\alpha \cap V$ and $\partial\mathcal{T}_\alpha = \partial\tilde{\mathcal{T}}_\alpha \cap V$.

We assume furthermore that we may take U_α small enough so that

(i) it is included in the domain of a local holomorphic chart $(z_1, \ldots, z_n)$ of W,

(ii) (the extensions $\tilde{E}$ and $\tilde{E}'$ of) the X-bundles E and E' of Lemmas 1 and 2 above are trivial on U_α.

We write

$$X|_{V_\alpha} = \sum_{i=1}^{n} A_i(z_1, \dots, z_n) \frac{\partial}{\partial z_i}.$$

Denote by $\mathcal{V}_i (1 \le i \le n)$ the open set of points m in $\partial \mathcal{T}_\alpha$ such that $A_i(m) \neq 0$. These open sets $\mathcal{V}_i$ constitute an open covering $\mathcal{V}$ of $\partial \mathcal{T}_\alpha$. Let $\mathcal{U}$ be any subcovering of V (for instance $\mathcal{V}$ itself; see also Theorem 2 in [**LS**]). We will denote by (R_i), $(1 \le i \le n)$ any sistem of "honey-comb cells" adapted to this covering $\mathcal{U}$ (see the definition in [**L**], section 1, under the name of "système d'alvéoles"). For instance, if the real hypersurfaces $|A_i| = |A_j| (i \neq j)$ in U_α are in general position, we may take for R_i the cell defined by: $|A_i| \ge |A_j|$ for all j, $j \neq i$, $\mathcal{V}_j \in \mathcal{U}$.

Denote by $\mathcal{M}(\mathcal{U})$ the set of $p - 1$ simplices in the "nerve" of $\mathcal{U}$, i.e., the set of all p-multiindices $u = (1 \le u_1 < u_2 < \cdots < u_p \le n)$ such that $\bigcap_{j=1}^{p} \mathcal{V}_{u_j}$ is not empty. For any $u \in \mathcal{M}(\mathcal{U})$, define

$$R_u = R_{u_1 u_2 \dots u_p} = \bigcap_{j=1}^{p} R_{u_j},$$

oriented as in section 1 of [**L**].

Let $\varphi \in (\mathbb{R}[c_1, \dots, c_r])^{2k}$ and $\varphi \in (\mathbb{R}[c_1, \dots, c_{r'}]^{2k'}$ be polynomials with $k + k' = p$ as in Lemmas 1 and 2 above. Taking a trivialization $(\sigma_1, \dots, \sigma_r)$ of $E|_{V_\alpha}$, we define the matrix M_α with holomorphic coefficients $(M_\alpha)_a^b \colon V_\alpha - \Sigma_\alpha \to \mathbb{C}$ by $\theta_{X_0}(\sigma_a) = \sum_b (M_\alpha)_a^b \sigma_b$. We define the matrix M'_α similarly for E'. Then we have the following.

Theorem 2.

$$I_{m_\alpha}(E, E', \varphi, \varphi', \theta_X, \theta'_X) = (-1)^{\left[\frac{p}{2}\right]} \sum_{u \in \mathcal{M}(\mathcal{U})} \int_{R_u} \frac{\varphi(M_\alpha)\varphi'(M'_\alpha) dz_{u_1} \wedge dz_{u_2} \wedge \ldots \wedge dz_{u_p}}{\prod_{j=1}^{p} A_{u_j}}.$$

We shall omit the proof, which is just a generalization of the proof of Theorem 1′ in [**LS**], corresponding to the absolute case $(k = p)$.

In particular, assume Σ_α consists of an isolated point m_α in V and that, near m_α, X is the restriction of a holomorphic vector field $\tilde{X}$ defined on a neighborhood of m_α in W. Then, after Theorem 2 of [**LS**], we have

Lemma 4. *There exists a local holomorphic chart* $(z_1, \ldots, z_n)$ *near* m_α *in* W*, such that the sequence* $(A_1, \ldots, A_{n-q}, f_1, \ldots, f_q)$ *is regular and hence* $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_p$ *cover* $\partial \mathcal{T}_\alpha (p = \dim_{\mathbb{C}} V)$*, where* $\tilde{X}$ *is expressed as*

$$\sum_{i=1}^{n} A_i(z_1, \ldots, z_n) \frac{\partial}{\partial z_i}$$

and $(f_1, \ldots, f_q)$ *denotes a system of defining functions of* V *near* m_α.

Corollary. *For a covering* $\mathcal{U}$ *as in Lemma 4,*

$$I_{m_\alpha}(E, E', \varphi, \varphi', \theta_X, \theta'_X) = \int_R \frac{\varphi(M_\alpha)\varphi'(M'_\alpha dz_1 \wedge dz_2 \wedge \ldots \wedge dz_p}{\prod\limits_{i=1}^{p} A_i},$$

where

$$R = \{z \| A_i(z)| = \varepsilon, \quad f_\lambda(z) = 0, \quad i = 1, \ldots, p, \quad \lambda = 1, \ldots, q\},$$

which is oriented so that the form $d\theta_1 \wedge \ldots \wedge d\theta_p$ *is positive,* $\theta_i = \arg A_i(z)$.

3. The Poincaré-Hopf index for vector fields on singular varieties

Let us apply the results of section 2 above, taking as E the restriction $T(W)|_V$ to V of the complex tangent bundle $T(W)$ of W and as E' the bundle ν_V, with the X actions θ_X and θ'_X already described in [**LS**]. Thus, with notation of Theorem 2 above, M_α is equal to $-J$, while $M'_\alpha = -C$. If we denote by $[c \cdot (c')^{-1}]_{n-q}$ the image of c''_{n-q} by the morphism

$$\mathbb{Z}[c''_1, \ldots, c''_{n-q}] \to \mathbb{Z}[c_1, \ldots, c_n] \otimes \mathbb{Z}[c'_1, \ldots, c'_q]$$

defined as in Lemma 3, then by Corollary to Theorem 2, the integral in Theorem 1 is equal to the index $I_{m_0}(T(W)|_V, \nu_V, [c \cdot (c')^{-1}]_{n-q}, \theta_X, \theta'_X)$ at an isolated point m_0 of the singular set Σ. (Here we assume that, near m_0, X is the restriction of a holomorphic vector field $\tilde{X}$ on a neighborhood of m_0 in W). Thus, Theorem 1 will follow from Lemma 2, if we prove the following Lemma 5. Recall that the topological index considered in [**GSV**] and [**SS**] coincides with the total index of a vector field obtained from X by perturbation on a smoothing of V.

Lemma 5. *The index* $I_{m_0}(TW|_V, \nu_V, [c\cdot(c')^{-1}]_{n-q}, \theta_X, \theta'_X)$ *coincides with the topological index of [**GSV**] and [**SS**]. In particular, if* m_0 *is a regular point of* V*, then it is equal to the usual Poincaré-Hopf index at* m_0 *of the vector field* X *on* V.

To prove the lemma, first let V be a complex manifold of dimension p and X or C^∞ section of $T(V)$ (C^∞ vector field on V) with singular set Σ. Let Σ_0 be a union of connected components of Σ and V_0 an open neighborhood of Σ_0 in V, not intersecting $\Sigma - \Sigma_0$. Assume that Σ_0 is compact and let $\mathcal{T}$ be a compact real manifold of dimension $2p$ with boundary, contained in V_0 and containing Σ_0 in its interior. Let ω''_0 be an arbitrary connection on $T(V)|_{V_0}$ and ω'' an X-trivial connection (i.e., a connection whose derivation law ∇ satisfies $\nabla X = 0$) on $T(V)|_{V_0 - \Sigma_0}$. Thus we have the vanishing of the characteristic form $\Delta_{\omega''}(c_p)$ on $V_0 - \Sigma_0$. We define the (topological) index $\mathrm{Ind}_{\Sigma_0}(X)$ of X at Σ_0 by

$$\mathrm{Ind}_{\Sigma_0}(X) = \int_{\mathcal{T}} \Delta_{\omega''_0}(c_p) + \int_{\partial\mathcal{T}} \Delta_{\omega''_0\omega''}(c_p).$$

Then it is not difficult to show that this index does not depend on the choice of the connection ω''_0 and the X-trivial connection ω''. Moreover, it is equal to the sum of the (topological) indices of a vector field on V obtained by perturbing X near Σ_0 so that it acquires only a finite number of isolated singular points.

Next we suppose that V is a (possibly) singular subvariety, which is an SLCI with a C^∞ extension $\tilde{\nu}_V$ of ν_V, in W and that X is a C^∞ vector field on the regular part V' of V with singular set Σ. Recall that there is a natural bundle map $\pi: T(W)|_V \to \nu_V$ which is surjective on V' with kernel $T(V')$. Thus we have the exact sequence

$$0 \to T(V') \to T(W)|_{V'} \to \nu_{V'} \to 0.$$

Let Σ_0, U_0 and $\mathcal{T}$ be as in Lemma 2 and set $V_0 = U_0 \cap V$. Also, let ω_0 and ω'_0 be arbitrary connections on $T(W)|_{U_0}$ and $\tilde{\nu}_V|_{U_0}$, respectively. Letting ω'' be an X-trivial connection on $T(V_0 - \Sigma_0)$, we take connections ω on $T(W)|_{V_0-\Sigma_0}$ and ω' on $\nu_{V_0-\Sigma_0}$ so that $(\omega'', \omega, \omega')$ is compatible, on

$V_0 - \Sigma_0$, with the above exact sequence. If we write

$$[c \cdot (c')^{-1}]_{n-q} = \sum_{i=0}^{r} \varphi_i \cdot \varphi'_i,$$

then we have the vanishing

$$\sum_{i=0}^{r} \Delta_\omega(\varphi_i) \wedge \Delta_{\omega'}(\varphi'_i) = \Delta_{\omega''}(c_p) = 0$$

on $V_0 - \Sigma_0$. We define the virtual (topological) index $v - \mathrm{Ind}_{\Sigma_0}(X)$ of X at Σ_0 to be equal to

$$\sum_{i=0}^{r} \Big(\int_T \Delta_{\omega_0}(\varphi_i) \wedge \Delta_{\omega'_0}(\varphi'_i) + \\ + \int_{\partial T} [\Delta_{\omega_0 \omega}(\varphi_i) \wedge \Delta_{\omega'_0}(\varphi'_i) + \Delta_\omega(\varphi_i) \wedge \Delta_{\omega'_0 \omega'}(\varphi'_i)] \Big).$$

Then it is not difficult to show that this index does not depend on the choice of the connections ω_0, ω'_0, ω'', ω, ω' with the above properties. Furthermore, if Σ_0 is in the regular part of V, then we have $v - \mathrm{Ind}_{\Sigma_0}(X) = \mathrm{Ind}_{\Sigma_0}(X)$.

Now we suppose that X is holomorphic and we choose the connections ω'', ω and ω' with the properties above so that, in addition, ω and ω' are special relative to θ_X and θ'_X, respectively. This is possible since θ_X leaves the sections of $T(V')$ invariant and θ'_X is defined from θ_X by passing to the quotient. Then we see that

$$v - \mathrm{Ind}_{\Sigma_0}(X) = I_{\Sigma_0}(TW|_V, \nu, [c \cdot (c')^{-1}]_{n-q}, \theta_X, \theta'_X).$$

To finish the proof of the lemma, let $\Sigma_0 = \{m_0\}$, U_0 a sufficiently small polydisk about m_0 and $\tilde{X}$ a holomorphic vector field on U_0 extending X. Also, let V be defined by $f: U_0 \to \mathbb{C}^q$ in U_0. There is an extension $\tilde{\pi}: T(W)|_{U_0} \to \tilde{\nu}_V|_{U_0}$ of $\pi|_{V_0}$ which is surjective away from the critical set C_f of f with kernel the bundle $T(f|_{U_0 - C_f})$ of vectors tangent to the fibers of f. Thus we have the exact sequence

$$0 \to T(f|_{U_0 - C_f}) \to T(W)|_{U_0 - C_f} \to \tilde{\nu}_V|_{U_0 - C_f} \to 0.$$

Let D be a one-dimensional disk about 0 in $\mathbb{C}^q$ such that $V_t = f^{-1}(t)$ is smooth for t in $D - \{0\}$. Thus, for t in $D - \{0\}$, V_t is in $U_0 - C_f$ and the restrictions $T(f|_{U_0-C_f})|V_t$ and $\tilde{\nu}_V|_{V_t}$ are, respectively, the tangent bundle $T(V_t)$ of V_t and the normal bundle ν_{V_t} of V_t in U_0. Starting from the holomorphic vector field $\tilde{X}$, we may construct a C^∞ section X' of $T(f|_{U_0-C_f})$ (on $U_0 - C_f$) so that it extends X (restricted to $V_0 - \{m_0\}$) and that the singular set Σ_t of the vector field $X_t = X'|_{V_t}$ on V_t is compact for all t near 0 in D. This is done by choosing a C^∞ splitting of the above sequence. We denote by $\tilde{\Sigma}$ the singular set of X' so that $\Sigma_t = \tilde{\Sigma} \cap V_t$. Let ω_0 and ω_0' be arbitrary connections on $T(W)|_{U_0}$ and $\tilde{\nu}_V|_{U_0}$, respectively. Letting $\tilde{\omega}''$ be an X'-trivial connection on $T(f|_{U_0-\tilde{\Sigma}})$, we take connections $\tilde{\omega}$ on $T(W)|_{U_0-\tilde{\Sigma}}$ and $\tilde{\omega}'$ on $\tilde{\nu}_{U_0-\tilde{\Sigma}}$ so that $(\tilde{\omega}'', \tilde{\omega}, \tilde{\omega}')$ is compatible, on $U_0 - \tilde{\Sigma}$, with the above exact sequence. For each t in D, we compute the virtual index $v - \mathrm{Ind}_{\Sigma_t}(X_t)$ using ω_0, ω_0' and the restrictions of $\tilde{\omega}''$, $\tilde{\omega}$, $\tilde{\omega}'$ to V_t. Then $v - \mathrm{Ind}_{\Sigma_t}(X_t)$ varies continuously on t. For $t \neq 0$, this is $\mathrm{Ind}_{\Sigma_t}(X_t)$, which is equal to the sum of the Poincaré-Hopf indices of a perturbation of X_t and for $t = 0$, this coincides with

$$I_{m_0}(T(W)|_V, \nu_V, [c \cdot (c')^{-1}]_{n-q}, \theta_X, \theta_X').$$

This proves the lemma.

Remark. If m_0 is a regular point of V, then, by Lemma 3, for an arbitrary polynomial φ of degree p, $I_{m_0}(TW|_V, \nu_V, \varphi, \theta_X, \theta_X')$ coincides with the residue of X for φ of Baum and Bott [**BB**$_1$] and [**BB**$_2$].

4. Examples

1) Assume $q = 1$, and f to be a quasi-homogeneous function of weight $(d_1, \ldots, d_n)$ from $\mathbb{C}^n$ into $\mathbb{C}$: this means that $X \cdot f = f$ (i.e. $g \equiv 1$), with

$$X = \sum_{i=1}^{n} \frac{z_i}{d_i} \frac{\partial}{\partial z_i}.$$

The formula of Theorem 1 for hypersurfaces becomes therefore

$$\mathrm{Ind}_{V,m_0}(X) = 1 + (-1)^{n-1} \prod_{i=1}^{n} (d_i - 1).$$

[Notice that the domain R of integration with respect to coordinates

$z_1, \dots, z_{n-1}$ is a covering with d_n sheets of the domain

$$|z_i| = \varepsilon, (i = 1 \dots, n-1):$$

this results from the fact that local coordinates have been choosen so that $(z_1, \dots, z_{n-1}, f)$ is a regular sequence.]

But X being transversal to the boundary of the Milnor fiber of f, $\mathrm{Ind}_{V,m_0}(X)$ is also equal to $1 + (-1)^{n-1}\mu(f)$, where $\mu(f)$ denotes the Milnor number: we recover the value

$$\prod_{i=1}^{n}(d_i - 1)$$

of $\mu(f)$ for quasi-homogeneous functions f (cf. [**A**]).

2) Assume $q = 2$, $f = (P, Q)$ to be given by homogeneous polynomials P and Q of respective degrees k, l, and take for X the vector field

$$H = \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i}$$

of infinitesimal homotheties: $\lambda_i = 1$ for any $i = 1 \dots, n$, while $\mu_1 = k$ and $\mu_2 = l$. The basis of $\mathbb{C}^n$ being choosen so that $(z_1, \dots, z_{n-2}, P, Q)$ is a regular sequence, the domain R of integration with respect to coordinates $z_1, \dots, z_{n-2}$ is a covering with kl sheets of the domain $|z_i| = \varepsilon$, $(i = 1 \dots, n-2)$. We get therefore:

$$\mathrm{Ind}_{V,m_0}(H) = kl \sum_{j=0}^{n-2} (-1)^j \binom{n}{n-2-j} \frac{k^{j+1} - l^{j+1}}{k - l}.$$

Of course, we recover the value 1 in the regular case $k = l = 1$.

3) In the situation of example 2, for $n = 4$, the previous formula gives

$$kl(6 - 4(k + l) + (k^2 + kl + l^2)).$$

If, instead of the residue of c_2, we take the residue of $(c_1)^2$, we get:

$$kl(16 - 6(k + l) + (k^2 + kl + l^2)).$$

4) Take $W = \mathbf{PC(3)}$ (the point of homogeneous coordinates X, Y, Z, T will be denoted $[X, Y, Z, T]$). Let V the complex quadratic cone of

equation $X^2 - YZ = 0$. Let λ be a complex parameter, and

$$X_\lambda = \lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + (2\lambda - 1) z \frac{\partial}{\partial z}$$

the vector field in the affine open set $T \neq 0$, with coordinates

$$x = \frac{X}{T}, y = \frac{Y}{T}, z = \frac{Z}{T}.$$

Notice that the point $[1,0,0,0]$ in the plane $T = 0$ does not belong to V, so that, near $T = 0$, it is enough to calculate in the affine spaces $Z \neq 0$ with coordinates

$$x' = \frac{X}{Z}, y' = \frac{Y}{Z}, t' = \frac{T}{Z},$$

and $Y \neq 0$ with coordinates

$$x'' = \frac{X}{Y}, z'' = \frac{Z}{Y}, t'' = \frac{T}{Y},$$

where X_λ may be written respectively

$$(1-\lambda) x' \frac{\partial}{\partial x'} + 2(1-\lambda) y' \frac{\partial}{\partial y'} + (1 - 2\lambda) t' \frac{\partial}{\partial t'},$$

and

$$(\lambda - 1) x'' \frac{\partial}{\partial x''} + 2(\lambda - 1) z'' \frac{\partial}{\partial z''} - t'' \frac{\partial}{\partial t''},$$

and may be extended near the points of V at infinity. For $\lambda \neq 0, \frac{1}{2}$, 1, the vector field X_λ has, on V, 3 singular points which are isolated: $m_1 = [0,0,0,1]$, $m_2 = [0,0,1,0]$ and $m_3 = [0,1,0,0]$. Using theorem 1, we get:

$$\mathrm{Ind}_{V,m_1}(X_\lambda) = 2, \mathrm{Ind}_{V,m_2}(X_\lambda) = 1, \quad \text{and} \quad \mathrm{Ind}_{V,m_3}(X_\lambda) = 1.$$

If we use now Theorem 2 for

$$c_1^2(T(\mathbf{PC(3)})|_V - \nu_V) = (c_1(T(\mathbf{PC(3)})|_V - c_1(\nu_V))^2,$$

we get the (now not necessarily integral) residues

$$I_{m_\alpha}(T(W)|_V, \nu_V, (c_1 - c_1')^2, \theta_X, \theta_X'),$$

respectively equal to

$$\frac{2\lambda^2}{2\lambda - 1}, \frac{(3\lambda - 2)^2}{(2\lambda - 1)(\lambda - 1)}, \quad \text{and} \quad -\frac{(\lambda - 2)^2}{\lambda - 1} \quad \text{for} \quad \alpha = 1, 2, 3.$$

Remark. If $\hat{L} \to V$ denotes the restriction to V of the tautological line bundle in hyperplanes on $\mathbf{PC(3)}$, then the restriction of $T(\mathbf{PC(3)})$ to V is stably equivalent to $4\hat{L}$, while the normal bundle ν_V is equal to $\hat{L}^{\otimes 2}$ since V has degree 2. Hence, the virtual tangent bundle to V is stably equivalent to $4\hat{L} - \hat{L}^{\otimes 2}$. Writing $\gamma = c_1(\hat{L})$, one gets the total Chern class

$$c(V) = c(4\hat{L} - \hat{L}^{\otimes 2}) = \frac{(1+\gamma)^4}{(1+2\gamma)}, c_1(V) = 2\gamma,$$

and $c_2(V) = 2\gamma^2$: since $\langle \gamma^2, [V] \rangle = 2$ (the degree of V), we recover: $\langle c_2(V), [V] \rangle = 4 = 2 + 1 + 1$, while

$$\langle c_1^2(V), [V] \rangle = 8 = \frac{2\lambda^2}{2\lambda - 1} + \frac{(3\lambda - 2)^2}{(2\lambda - 1)(\lambda - 1)} - \frac{(\lambda - 2)^2}{\lambda - 1}$$

(in particular, the sum of these residues is an integer, while in general none of them is).

This example has also been studied locally near m_1 in [**BG**], for showing the upper semicontinuity at $\lambda = 0$ of the multiplicity of X_λ on V at m_1.

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